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Tensor Analysis and Curvature in Quantum Space-Time

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Introducing quantum space-time into physics by means of the transformation language of noncommuting coordinates gives a simple scheme of generalizing the tensor analysis. The general covariance principle for the quantum space-time case is discussed, within which one can obtain the covariant structure of basic tensor quantities and the motion equation for a particle in a gravitational field. Definitions of covariant derivatives and curvature are also generalized in the given case. It turns out that the covariant structure of the Riemann-Christoffel curvature tensor is not preserved in quantum space-time. However, if the curvature tensor $\hat{R}_{\mu\nu\lambda\varkappa}(z)$ is redetermined up to the value of the L^2 term, then its covariant structure is achieved, and it, in turn, allows us to reconstruct the Einstein equation in quantum space-time.

1. INTRODUCTION

In a previous paper (Namsrai, 1986) we have shown that due to an additional force caused by quantum space-time structure the equivalence principle between gravity and inertia is achieved up to $O(L^2)$, where L is the fundamental length. There we applied the equivalence principle in order to introduce the gravitational effect into physical systems in the case of quantum space-time. Following this, we wrote down equations in a virtual "quasilocal" inertial system of coordinates [i.e., equations of the special theory of relativity such that $d^2\xi^\alpha/d\tau^2=f^\alpha(\xi)$, where $f^\alpha(\xi)$ is an additional force proportional to the L^2 term] and next carried out a transformation of coordinates in order to find corresponding equations in a quantum system of reference.

While this method could be used further, here we employ another method [for details, see Weinberg (1972)], which has the same physical

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content, but is more elegant in its notation and more convenient to handle. This approach is based on the alternative version of the equivalence principle known as the principle of general covariance. It asserts that a physical equation is given in an arbitrary gravitational field in the case where the following two conditions are fulfilled:

1. The equation is given in the absence of gravity, i.e., it corresponds to the laws of the special theory of relativity [in our case it is slightly modified according to Namsrai (1986)] when its metric tensor $g_{\alpha\beta}$ is equal to Minkowski's $\eta_{\alpha\beta}$ and the affine connection $\Gamma^{\nu}_{\lambda\mu}(x)$ disappears.

2. The equation is generally covariant, i.e., it preserves its form under an arbitrary transformation of coordinates $x^{\nu} \to x'^{\nu}$.

It should be noted that, as shown below, in quantum space-time one can obtain the covariant form of the motion equation for a particle in a gravitational force. However, we do not succeed in preserving the covariant structure of the curvature tensor in quantum space-time and it, in turn, gives rise to the reformulation of the general covariance principle up to the order of the L^2 term for the Einstein equation case. Moreover, in the quantum system of reference

$$x^{\mu} \to x'^{\mu} \equiv z^{\mu} = x^{\mu} + L\Pi^{\mu}(x) \tag{1}$$

[$\Pi^{\mu}(x)$ are arbitrary noncommutative functions] tensor algebra is more restricted with respect to the usual space-time transformation of c-number coordinates.

This paper is an immediate continuation of an earlier work (Namsrai, 1986) and is devoted to the study of tensor algebra under the transformation (1). In Section 2 we present some mathematical peculiarities of our scheme and define left-hand and right-hand derivatives of any function $f(z)$ depending on quantum variables z^{μ}. Section 3 deals with tensor algebra with respect to the transformation of quantum coordinates (1). The requirement of the covariant structure of the motion equation of a particle in the gravitational force gives a unique form to the affine connection in quantum space-time. This problem and transformation of the affine connection are presented in Section 4. In Sections 5 and 6 we generalize the definition of covariant derivative and its form along a given curve $z^{\mu}(\tau)$. Sections 7 and 8 are devoted to the definition of the curvature tensor and to the reconstruction of the Einstein equation in the quantum space-time case, respectively.

2. THE MATHEMATICAL PECULIARITY OF THE QUANTUM TRANSFORMATION OF COORDINATE SYSTEM

We suggest that in the microworld space-time $R^4(z^{\mu})$ may possess some quantum nature and therefore physical quantities depend on noncommuting variables z^μ;

$$[z^\mu, z^\nu]_- \neq 0 \qquad \text{for} \quad \mu \neq \nu$$

at small distances. Further, it is necessary to pass to a large scale in order to construct a physical theory in real nonquantum space-time $R^4(x^\nu)$. This passage could be carried out by means of the transformation language of coordinate systems x^μ and z^μ if we could construct a transformationn law of physical quantities under the transformation $x^\mu \to z^\mu$ or $z^\mu \to x^\mu$. Thus, our final aim is to study the residual effect or contribution caused by this transformation to any physical processes and quantities on a large scale.

In order to find the transformation law of physical quantities under the transformation $x^\mu \rightleftarrows z^\mu$, we should first give the definition of transformation matrices $\partial x^\rho/\partial z^\mu$ or $\partial z^\nu/\partial x^\delta$ and their products of the type

$$\frac{\partial x^\rho}{\partial z^\mu}\frac{\partial z^\mu}{\partial x^\delta}, \qquad \frac{\partial z^\nu}{\partial x^\delta}\frac{\partial x^\delta}{\partial z^\mu}, \ldots \tag{2}$$

and also differentiation of function $f(z)$ with respect to noncommuting variables z^μ. It is easily seen that a definition of the type (2) follows from the differentiation rule of $f(z^\mu)$. For this purpose, we define left-hand and right-hand derivatives of any function $f(z)$ with respect to z. By definition

$$\frac{\vec{\partial}}{\partial z^\mu} f(z) = \frac{\partial x^q}{\partial z^\mu}\frac{\vec{\partial}}{\partial x^q} f(z), \qquad f(z)\frac{\overleftarrow{\partial}}{\partial z^\mu} = \frac{\overleftarrow{\partial}}{\partial x^q} f(z)\frac{\partial x^q}{\partial z^\mu} \tag{3}$$

On the other hand,

$$\frac{\vec{\partial}}{\partial x^q} f(z) = \frac{\partial z^\rho}{\partial x^q}\frac{\partial f(z)}{\partial z^\rho}, \qquad \frac{\overleftarrow{\partial}}{\partial x^q} f(z) = \frac{\partial f(z)}{\partial z^\rho}\frac{\partial z^\rho}{\partial x^q}$$

It is natural to assume

$$\frac{\vec{\partial}}{\partial z^\mu} f(z) \equiv \frac{\partial}{\partial z^\mu} f(z) \qquad \text{or} \qquad \frac{\vec{\partial}}{\partial x^q} f(z) \equiv \frac{\partial}{\partial x^q} f(z)$$

which, in turn, gives the following equality:

$$\frac{\partial x^\alpha}{\partial z^\mu}\frac{\partial z^\lambda}{\partial x^\alpha} = \delta^\lambda_\mu \qquad \text{or} \qquad \frac{\partial z^\rho}{\partial x^\beta}\frac{\partial x^\delta}{\partial z^\rho} = \delta^\delta_\beta \tag{4}$$

To calculate an inverse operation $\partial x^\beta/\partial z^\mu$ of $\partial z^\mu/\partial x^\delta$, an explicit dependence between z^μ and x^μ should be given. For the simpler case, it is just (1). Since

$$\frac{\partial z^\mu}{\partial x^\alpha} = \delta^\mu_\alpha + L\frac{\partial \Pi^\mu}{\partial x^\alpha}$$

the form

$$\frac{\partial x^\alpha}{\partial z^\mu} = \delta^\alpha_\mu - L\frac{\partial \Pi^\alpha}{\partial x^\mu} + L^2\frac{\partial \Pi^\rho}{\partial x^\mu}\frac{\partial \Pi^\alpha}{\partial x^\rho} - L^3\frac{\partial \Pi^\rho}{\partial x^\mu}\frac{\partial \Pi^\delta}{\partial x^\rho}\frac{\partial \Pi^\alpha}{\partial x^\delta} + \cdots \tag{5}$$

satisfies conditions (4) automatically for any order of L. With this definition, the other type of product of transformation matrices takes the form

$$\frac{\partial x^\lambda}{\partial z^\alpha}\frac{\partial z^\alpha}{\partial x^\mu} \equiv \frac{\partial z^\lambda}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial z^\mu} = \delta^\lambda_\mu + LI^{\rho\lambda}_{\mu\rho} \tag{6}$$

where

$$I^{\rho\lambda}_{\mu\rho} = \left[\frac{\partial \Pi^\rho}{\partial x^\mu}, \frac{\partial \Pi^\lambda}{\partial x^\rho}\right]_-$$

Thus,

$$\frac{\vec{\partial}}{\partial z^\beta} f(z) - f(z)\frac{\overleftarrow{\partial}}{\partial z^\beta} = -L^2 \frac{\partial f(x)}{\partial x^\rho} I^{q\rho}_{\beta q} \tag{7}$$

We see that expressions (3)–(7) differ essentially from the usual transformation case ($x^\mu \to x'^\mu$), for which

$$\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x'^\lambda}{\partial x^\alpha} = \frac{\partial x'^\lambda}{\partial x^\beta}\frac{\partial x^\beta}{\partial x'^\mu} = \delta^\lambda_\mu$$

Notice that definition (4) indicates how to introduce the formal procedure of inserting the transformation matrix $\partial z^\rho/\partial x^\delta$ into any mathematical expression. For example, let x^λ and ξ^β be two c-number variables and $\partial x^\lambda/\partial \xi^\beta$ the transformation matrix between them. Then the latter may be written in two different forms:

$$\begin{gathered}\frac{\partial x^\lambda}{\partial \xi^\beta} = \frac{\partial x^\lambda}{\partial z^\rho}\frac{\partial z^\rho}{\partial \xi^\beta} = \left(\frac{\partial x^\lambda}{\partial z^\rho}\frac{\partial z^\rho}{\partial x^n}\right)\frac{\partial x^n}{\partial \xi^\alpha} \\ \frac{\partial x^\lambda}{\partial \xi^\beta} = \frac{\partial z^\rho}{\partial \xi^\beta}\frac{\partial x^\lambda}{\partial z^\rho} = \frac{\partial z^\rho}{\partial x^n}\frac{\partial x^n}{\partial \xi^\beta}\frac{\partial x^\lambda}{\partial z^\rho} = \left(\frac{\partial z^\rho}{\partial x^n}\frac{\partial x^\lambda}{\partial z^\rho}\right)\frac{\partial x^n}{\partial \xi^\beta}\end{gathered} \tag{8}$$

In accordance with definition (4), only the last form is acceptable. Thus, the change of differentiation variables is carried out by the following rule for any function $f(x(z))$:

$$\frac{\partial f}{\partial x^\mu} = \frac{\partial z^\rho}{\partial x^\mu}\frac{\partial f}{\partial z^\rho}, \qquad \frac{\partial f}{\partial z^\rho} = \frac{\partial x^n}{\partial z^\rho}\frac{\partial}{\partial x^n} f$$

where z^μ and x^μ are quantum and c-number variables, respectively. Following this rule, differentiation of the product of functions is defined as follows.

Let $G(z) = f_1(z) f_2(z)$; then

$$\frac{\partial G(z)}{\partial z^\mu} \overset{Df}{=} \frac{\partial x^n}{\partial z^\mu} \frac{\partial}{\partial x^n}[f_1(z) f_2(z)]$$

$$= \frac{\partial x^n}{\partial z^\mu}\left[\frac{\partial f_1(z)}{\partial x^n} f_2(z) + f_1(z)\frac{\partial f_2(z)}{\partial x^n}\right]$$

Now it is necessary to find the commutator $[\partial x^n/\partial z^\mu, f_1(z)]_-$. For this, taking into account (1) and (5) and making use of the so-called "Sylvester expansion" for the matrix function (see Frazer *et al.*, 1952) as well as the obvious power series expansion

$$f_1(x + L\Pi) = f_1(x) + L\Pi^\mu \frac{\partial f_1(x)}{\partial x^\mu} + \frac{L^2}{2!}\langle \Pi^\mu \Pi^\nu \rangle \frac{\partial^2 f_1(x)}{\partial x^\nu\, \partial x^\mu}$$

$$+ L^3/3!\langle \Pi^\mu \Pi^\nu \Pi^\rho \rangle \frac{\partial^3 f_1(x)}{\partial x^\mu\, \partial x^\nu\, \partial x^\rho}$$

where

$$\langle \Pi^{\mu_1} \Pi^{\mu_2} \cdots \Pi^{\mu_n} \rangle = \frac{1}{n!} \sum \Pi^{\mu_1} \Pi_2^{\mu} \cdots \Pi^{\mu_n}$$

the sum being taken over all the $n!$ permutations of the indices, and after some calculation, we have

$$\left[\frac{\partial x^n}{\partial z^\mu}, f_1(z)\right]_- = L^2\left[f_1(x) I^{\rho n}_{\mu\rho} + \frac{\partial f_1(x)}{\partial x^\nu}\left(\Pi^\nu \frac{\partial \Pi^n}{\partial x^\mu} - \frac{\partial \Pi^n}{\partial x^\mu}\Pi^\nu\right)\right] + O(L^3) \qquad (9)$$

Thus,

$$\frac{\partial G(z)}{\partial z^\mu} = \frac{\partial f_1(z)}{\partial z^\mu} f_2(z) + f_1(z)\frac{\partial f_2(z)}{\partial z^\mu} + \left[\frac{\partial x^n}{\partial z^\mu}, f_1(z)\right]_- \frac{\partial f_2(z)}{\partial x^n}$$

In the last term of this expression we must insert (9) and take $\partial f_2(z)/\partial x^n \sim \partial f_2(x)/\partial x^n$ by our level of accuracy. As a result we obtain the usual rule of differentiation of the product of functions up to $O(L^2)$.

Finally, we notice that in our scheme left-hand and right-hand products also give different results. For example, let the rule of taking the partial derivative be (see Section 3.1)

$$dz^\mu = \frac{\partial z^\mu}{\partial x^\nu}\, dx^\nu$$

Then multiply this by $\partial x^\rho/\partial z^\mu$ on the left- and right-hand sides, and we have, in accordance with (4) and (6),

$$\frac{\partial x^\rho}{\partial z^\mu} dz^\mu = \frac{\partial x^\rho}{\partial z^\mu} \frac{\partial z^\mu}{\partial x^\nu} dx^\nu = dx^\rho + L^2 I^{n\rho}_{\nu n} dx^\nu$$

and

$$dz^\mu \frac{\partial x^\rho}{\partial z^\mu} = dx^\nu \frac{\partial z^\mu}{\partial x^\nu} \frac{\partial x^\rho}{\partial z^\mu} = dx^\rho$$

respectively. Thus,

$$\left[dz^\mu, \frac{\partial x^\rho}{\partial z^\mu} \right]_- = L^2 I^{\rho n}_{n\nu} dx^\nu$$

where we have taken into account

$$I^{n\rho}_{\nu n} = \frac{\partial \Pi^n}{\partial x^\nu} \frac{\partial \Pi^\rho}{\partial x^n} - \frac{\partial \Pi^\rho}{\partial x^n} \frac{\partial \Pi^n}{\partial x^\nu} = -I^{\rho n}_{n\nu}$$

3. TENSOR ANALYSIS

3.1. Vectors and Tensors

To construct invariant physical equations with respect to a quantum transformation of coordinates we must know how quantities standing in equations under this transformation behave. We start from simpler physical quantities, such as vectors and tensors. By definition, as in the usual case, under the change of variables $x^\mu \to z^\mu$ contra- and covariant vectors V^μ and U_μ transform by the formulas

$$\hat{V}^\mu(z) = V^\nu(x) \frac{\partial z^\mu}{\partial x^\nu}, \qquad \hat{U}_\mu(z) = \frac{\partial x^\nu}{\partial z^\mu} U_\nu(x) \tag{10}$$

respectively. For example, the rule of taking the partial derivative gives

$$dz^\mu = \frac{\partial z^\mu}{\partial x^\nu} dx^\nu$$

so that the differential of coordinates is a contravariant vector. If ϕ is a scalar field, the $\partial\phi/\partial x^\mu$ is a covariant vector, since

$$\frac{\partial \phi}{\partial z^\mu} = \frac{\partial x^\nu}{\partial z^\mu} \frac{\partial \phi}{\partial x^\nu}$$

In order to find the transformation law of high-rank tensors, the definite sequence of their tensor indices should be indicated. For example, if $T_\nu^{\mu\lambda}$ is a tensor of the type of $U_\nu V^\mu W^\lambda$, then its transformation is given by

$$\hat{T}_\nu^{\mu\lambda}(z) = \frac{\partial x^\rho}{\partial z^\nu} \frac{\partial z^\mu}{\partial x^\varkappa} \frac{\partial z^\lambda}{\partial x^\delta} T_\rho^{\varkappa\delta}(x)$$

The requirement of strict arrangement of tensor indices is connected with the noncommutability properties of transformation matrices:

$$\left[\frac{\partial z^\rho}{\partial x^\nu}, \frac{\partial z^\delta}{\partial x^\mu}\right]_- = \left[\frac{\partial x^\rho}{\partial z^\nu}, \frac{\partial x^\delta}{\partial z^\mu}\right]_- = -\left[\frac{\partial x^\rho}{\partial z^\nu}, \frac{\partial z^\delta}{\partial x^\mu}\right]_- = L^2 I_{\nu\mu}^{\rho\delta} \tag{11}$$

The more important tensor is the metric tensor defined by the formula

$$g_{\mu\nu}(x) = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\nu} \frac{\partial \xi^\beta}{\partial x^\mu}$$

in an arbitrarily chosen system of reference x^μ. In a quantum system of coordinates z^μ the metric tensor reads

$$\hat{g}_{\mu\nu}(z) = \eta_{\alpha\beta} \frac{\partial \xi^\beta}{\partial z^\mu} \frac{\partial \xi^\alpha}{\partial z^\nu} = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\rho} \frac{\partial \xi^\beta}{\partial x^\delta} \frac{\partial x^\rho}{\partial z^\mu} \frac{\partial x^\delta}{\partial z^\nu}$$

and therefore

$$\hat{g}_{\mu\nu}(z) = g_{\rho\sigma}(x) \frac{\partial x^\rho}{\partial z^\mu} \frac{\partial x^\sigma}{\partial z^\nu}$$

from which we see that $\hat{g}_{\mu\nu}(z)$ is indeed the covariant tensor. An inverse tensor with respect to $\hat{g}_{\mu\nu}(z)$ is given by the relations

$$\hat{g}^{\mu\nu}(z)\hat{g}_{\nu\rho}(z) = \hat{g}_{\nu\rho}(z)\hat{g}^{\rho\mu}(z) = \delta_\nu^\mu \tag{12a}$$

where

$$\hat{g}^{\rho\mu}(z) = g^{kl}(x) \frac{\partial z^\rho}{\partial x^k} \frac{\partial z^\mu}{\partial x^l} - L^2 g^{k\mu}(x) I_{kq}^{q\rho} \tag{12b}$$

In accordance with definition (4) in the quantum space-time case the Kronecker symbol δ_μ^ν is a mixed tensor of the type $\hat{T}_\mu^\nu = \hat{U}_\mu(z)\hat{V}^\nu(z)$, but not $\hat{V}^\mu(z)\hat{U}_\nu(z)$; indeed,

$$\hat{\delta}_\mu^\nu = \frac{\partial x^q}{\partial z^\mu} \frac{\partial z^\nu}{\partial x^\rho} \delta_q^\rho = \delta_\mu^\nu$$

is invariant. In this connection, it should be noted that the invariant combination of the product of two vectors $\hat{V}^\mu$ and $\hat{U}_\mu$ is defined as

$$\hat{V}^\mu(z)\hat{U}_\mu(z) = \text{invariant}$$

But $\hat{U}_\mu(z)\hat{V}^\mu(z)$ is not invariant, so that

$$[\hat{V}^\mu(z), \hat{U}_\mu(z)]_- = -L^2 I_{\nu n}^{n\rho} U_\rho(x) V^\nu(x) \tag{13}$$

3.2. Tensor Algebra

In a quantum system of reference z tensor algebra is more limited with respect to the usual space-time transformation. Summation and product properties of the tensor transformation are preserved. Indeed, let $\hat{A}^{\mu}_{\nu}$ and $\hat{B}^{\mu}_{\nu}$ be two mixed tensors. Consider their sum $\hat{T}^{\mu}_{\nu} = a\hat{A}^{\mu}_{\nu} + b\hat{B}^{\mu}_{\nu}$ for any scalar constants a and b. Then $\hat{T}^{\mu}_{\nu}(z)$ is a tensor since

$$\hat{T}^{\mu}_{\nu}(z) = a\hat{A}^{\mu}_{\nu}(z) + b\hat{B}^{\mu}_{\nu}(z) = a\frac{\partial z^{\mu}}{\partial x^{\rho}}\,\partial x^{\sigma}/\partial z^{\nu} A^{\rho}_{\sigma}(x) + b\frac{\partial z^{\mu}}{\partial x^{\rho}}\frac{\partial x^{\sigma}}{\partial z^{\nu}} B^{\rho}_{\sigma}(x)$$
$$= \frac{\partial z^{\mu}}{\partial x^{\rho}}\frac{\partial x^{\sigma}}{\partial z^{\nu}} T^{\rho}_{\sigma}(x)$$

where it is assumed that tensor indices for $\hat{A}^{\mu}_{\nu}$ and $\hat{B}^{\mu}_{\nu}$ are arranged in a definite sequence, namely μ, ν. Moreover, for example, if $\hat{A}^{\mu}_{\nu}$ and $\hat{B}^{\mu}$ are tensors, the combination $T^{\mu\rho}_{\nu}(z) = \hat{A}^{\mu}_{\nu}(z)B^{\rho}(z)$ is also a tensor, i.e.,

$$\hat{T}^{\mu\rho}_{\nu}(z) \equiv \hat{A}^{\mu}_{\nu}(z)\hat{B}^{\rho}(z) = \frac{\partial z^{\mu}}{\partial x^{\lambda}}\frac{\partial x^{\varkappa}}{\partial z^{\nu}} A^{\lambda}_{\varkappa}(x)\frac{\partial z^{\rho}}{\partial x^{\sigma}} B^{\sigma}(x)$$
$$= \frac{\partial z^{\mu}}{\partial z^{\lambda}}\frac{\partial x^{\varkappa}}{\partial z^{\nu}}\frac{\partial z^{\rho}}{\partial x^{\sigma}} T^{\lambda\sigma}_{\varkappa}(x)$$

where a definite sequence of indices μ, ν, ρ is assumed.

Generally speaking, in the quantum transformation case any operations of contraction, lowering, and raising of the index for tensors do not lead to new tensors. Here we indicate some specific possibilities.

1. Let $\hat{T}^{\mu\rho\sigma}_{\nu}(z)$ be a tensor with definite arranged indices $\mu\rho\sigma\nu$ and if we obtain a tensor by means of the contraction operation of indices σ and ν, then $\hat{T}^{\mu\rho} \equiv T^{\mu\rho\nu}_{\nu}$ is also a tensor; indeed,

$$\hat{T}^{\mu\rho}(z) \equiv \hat{T}^{\mu\rho\nu}_{\nu}(z) = \frac{\partial z^{\mu}}{\partial x^{\varkappa}}\frac{\partial z^{\rho}}{\partial x^{\sigma}}\frac{\partial z^{\nu}}{\partial x^{\tau}}\frac{\partial x^{2}}{\partial z^{\nu}} T^{\varkappa\sigma\tau}_{\eta}(x)$$
$$= \frac{\partial z^{\mu}}{\partial x^{\varkappa}}\frac{\partial z^{\rho}}{\partial x^{\sigma}} T^{\varkappa\sigma\lambda}_{\lambda} = \frac{\partial z^{\mu}}{\partial x^{\varkappa}}\frac{\partial z^{\rho}}{\partial x^{\eta}} T^{\varkappa\eta} \tag{14}$$

2. The following two types of lowering and raising indices of a tensor preserve the tensor structure in the case of quantum space-time transformation:

$$\hat{S}^{\rho}_{\sigma\mu}(z) = \hat{T}^{\rho\lambda}_{\sigma}(z)\hat{g}_{\lambda\mu}(x) \qquad \text{for } \sigma\rho\lambda \text{ or } \rho\sigma\lambda \text{ sequences}$$
$$\hat{R}^{\nu\rho}_{\sigma}(z) = \hat{g}^{\nu\mu}_{1}(z)\hat{S}^{\rho}_{\mu\sigma}(z) \qquad \text{for } \mu\rho\sigma \text{ or } \mu\sigma\rho \text{ sequences} \tag{15}$$

where

$$\hat{g}^{\nu\mu}_{1}(z) = g^{\alpha\beta}(x)\frac{\partial z^{\nu}}{\partial x^{\alpha}}\frac{\partial z^{\mu}}{\partial x^{\beta}}$$

Then $\hat{S}^{\rho}_{\sigma\mu}(z)$ and $\hat{R}^{\nu\rho}_{\sigma}(z)$ are tensors. In expressions (14) and (15) we have used definition (4).

3. According to (12a), raising and lowering of both indices for the metric tensor $\hat{g}_{\mu\nu}(z)$ are carried out by the following rules:

$$\hat{g}^{\lambda\mu}(z)\hat{g}^{\varkappa\nu}(z)\hat{g}_{\nu\mu}(z)=\hat{g}^{\lambda\mu}(z)\hat{g}_{\mu\nu}(z)\hat{g}^{\nu\varkappa}(z)=\hat{g}^{\lambda\mu}(x)\delta^{\varkappa}_{\mu}=\hat{g}^{\lambda\varkappa}(z)$$

and

$$\hat{g}_{\lambda\mu}(z)\hat{g}^{\varkappa\nu}(z)\hat{g}_{\nu\mu}(z)=\hat{g}_{\lambda\mu}(z)\hat{g}_{\mu\nu}(z)\hat{g}^{\nu\varkappa}(z)=\hat{g}_{\lambda\mu}(z)\delta^{\varkappa}_{\mu}=\hat{g}_{\lambda\varkappa}(z)$$

This specific rule of lowering and raising indices for $g_{\mu\nu}(z)$ again gives the metric tensor and its inverse, respectively.

3.3. Tensor Density

An important example of nontensor values is the determinant of the metric tensor

$$\hat{g}=-\text{Det }\hat{g}_{\mu\nu}(z)$$

The rule of the metric tensor transformation may be regarded as a matrix equation

$$\hat{g}_{\mu\nu}(z)=\frac{\partial x^{\rho}}{\partial z^{\mu}}g_{\rho\sigma}(x)\frac{\partial x^{\sigma}}{\partial z^{\nu}}$$

Calculating its determinant, we have

$$\hat{g}=|\partial x/\partial z|^{2}g \tag{16}$$

where $|\partial x/\partial z|$ is the Jacobian of the transformation $z^{\mu}\to x^{\mu}$, i.e., the determinant of the matrix $\partial x^{\rho}/\partial z^{\mu}$. As in the usual case, if we do not take into account an additional multiplier caused by the Jacobian, we call a quantity of the type of $\hat{g}$ a scalar density in the quantum system of reference z^{μ}. Similarly, a value that transforms as a tensor but with additional multipliers from the Jacobian is called a tensor density. We call the number of factors $|\partial z/\partial x|$ in the determinant the weight of the density. For example, from expression (16) it follows that $\hat{g}$ is a density with weight -2 up to $O(L^8)$, since

$$|\partial x/\partial z|=|\partial z/\partial x|^{-1}+O(L^{8})$$

The latter is easily verified by estimating the determinant of the equation

$$\frac{\partial x^{\mu}\partial z^{\lambda}}{\partial z^{\lambda}\partial x^{\nu}}=\delta^{\mu}_{\nu}+L^{2}I^{n\mu}_{\nu n}$$

Any tensor density weight w can be expressed as a usual tensor multiplied by the coefficient $g^{-w/2}$. For example, the tensor density F^{μ}_{ν} with weight w transforms by the rule

$$\hat{F}^{\mu}_{\nu}(z)=\left|\frac{\partial z}{\partial x}\right|^{w}\frac{\partial z^{\mu}}{\partial x^{\lambda}}\frac{\partial x^{\varkappa}}{\partial z^{\nu}}F^{\lambda}_{\varkappa}(x)$$

Using (16), we find

$$\hat{g}^{w/2}\hat{F}^{\mu}_{\nu}(z)=\frac{\partial z^{\mu}}{\partial x^{\lambda}}\frac{\partial x^{\varkappa}}{\partial z^{\nu}}g^{w/2}F^{\lambda}_{\varkappa}(x)$$

An important role of tensor densities is defined by the fundamental theorem of integral calculus, which asserts that under an arbitrary transformation of coordinates $x^{\mu}\to x'^{\mu}=z^{\mu}$ the volume element d^4x is replaced by

$$d^4z=|dz/dx|\,d^4x$$

In our case, $d^4z=d^4x\,[1+O(L^4)]$, and therefore the product of d^4x on the tensor density with the weight -1 transforms as a usual tensor. In particular, $\hat{g}^{1/2}\,d^4z\sim g^{1/2}\,d^4x$ is an invariant element of the volume.

4. MOTION EQUATION OF THE PARTICLE AND TRANSFORMATION OF THE AFFINE CONNECTION IN QUANTUM SPACE-TIME

4.1. Covariant Structure of Motion Equation

In accordance with the "slightly violated principle of equivalence" formulated in Namsrai (1986), there exists a "freely" falling system of reference ξ^{α} in which a particle moves along an almost rectilinear trajectory given by the equation

$$\frac{d^2\xi^{\alpha}}{d\tau^2}=\frac{1}{m}f^{\alpha}(\xi) \tag{17}$$

where $f^{\alpha}(\xi)$ and

$$d\tau^2=\eta_{\alpha\beta}\,d\xi^{\alpha}\,d\xi^{\beta} \tag{18}$$

are some averaged "quantum" force proportional to the L^2-term and the proper time, respectively. $\eta_{\alpha\beta}$ is the Minkowski metric. Further, we assume that we take a curvilinear quantum system of reference z^{μ} connected with the usual curvilinear one x^{μ} by relation (1). In the usual case, when the coordinates ξ^{α} of "freely" falling system of reference are functions of x^{μ} only, equation (17) is transformed into the well-known form

$$\frac{d^2x^{\lambda}}{d\tau^2}+\Gamma^{\lambda}_{\mu\nu}(x)\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}=\frac{1}{m}f^{\lambda}(x) \tag{19}$$

where the proper time is given by

$$d\tau^2 = g_{nm}(x)\, dx^n\, dx^m, \qquad g_{nm}(x) = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^n} \frac{\partial \xi^\beta}{\partial x^m} \tag{20a}$$

and

$$\Gamma^\lambda_{\nu\mu}(x) = \frac{\partial x^\lambda}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha}{\partial x^\nu\, \partial x^\mu} \tag{20b}$$

is called the affine connection. Here

$$f^\lambda(x) = \frac{\partial x^\lambda}{\partial \xi^\alpha} f^\alpha(\xi)$$

In a quantum system of reference z^μ the coordinates ξ^α are functions of z^μ and x^μ; then equation (17) acquires the form

$$\frac{d^2 z^\mu}{d\tau^2} \frac{\partial \xi^\alpha}{\partial z^\mu} + \frac{dz^\mu}{d\tau} \frac{dz^\nu}{d\tau} \frac{\partial^2 \xi^\alpha}{\partial z^\nu\, \partial z^\mu} = \frac{1}{m} f^\alpha(\xi) \tag{21}$$

where we have used the definition

$$\frac{\vec{d}\xi^\alpha}{d\tau} = \frac{dz^\mu}{d\tau} \frac{\partial \xi^\alpha}{\partial z^\mu}$$

Multiplying equation (21) on the right-hand side by $\partial z^\lambda / \partial \xi^\alpha$ and making use of (4), we get

$$\frac{d^2 z^\lambda}{d\tau^2} + \frac{dz^\mu}{d\tau} \frac{dz^\nu}{d\tau} \hat{\Gamma}^\lambda_{\nu\mu}(z) = \frac{1}{m} f^\lambda(z) \tag{22}$$

where

$$\hat{\Gamma}^\lambda_{\nu\mu}(z) = \frac{\partial^2 \xi^\alpha}{\partial z^\nu\, \partial z^\mu} \frac{\partial z^\lambda}{\partial \xi^\alpha} \tag{23}$$

generalizes the definition of the affine connection $\Gamma^\lambda_{\nu\mu} x)$ given by (20b) in the usual theory. The "quantum" force in (22) is defined in a natural manner:

$$f^\lambda(z) = \frac{\partial z^\lambda}{\partial \xi^\alpha} f^\alpha(\xi) = \frac{\partial z^\lambda}{\partial x^\rho} f^\rho(x)$$

in accordance with definition (10).

The proper time (18) may also be expressed in a quantum system of reference,

$$d\tau^2 = \eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial z^\mu}\, dz^\mu \frac{\partial\xi^\beta}{\partial z^\nu}\, dz^\nu$$

$$= g_{nm}(x)\frac{\partial x^n}{\partial z^\mu}\frac{\partial x^m}{\partial z^\nu}\, dz^\mu\, dz^\nu$$

$$+ L^2 g_{nm}(x)\frac{\partial x^n}{\partial z^\mu} I^{m\mu}_{\nu\rho}\, dx^\rho\, dz^\nu \tag{24}$$

where we have used the following value of the commutator:

$$[dz^\mu, \partial x^m/\partial z^\nu]_- = L^2 I^{m\mu}_{\nu\rho}\, dx^\rho \tag{25}$$

and the usual definition (20a) for $g_{nm}(x)$. It is easily verified that the last term in (24) is equal to zero up to $O(L^3)$ term. Thus,

$$d\tau^2 = \hat{g}_{\mu\nu}(z)\, dz^\mu\, dz^\nu \tag{26}$$

where

$$\hat{g}_{\mu\nu}(z) = g_{nm}(x)\frac{\partial x^n}{\partial z^\mu}\frac{\partial x^m}{\partial z^\nu} = \eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial z^\mu}\frac{\partial\xi^\beta}{\partial z^\nu} \tag{27}$$

is the metric tensor in quantum space-time. So, we see that the covariant structure of the proper time and the metric tensor is preserved in our formalism.

It is interesting to notice that if we use the transformation of the differential in the product form

$$dx^n = dz^\mu\frac{\partial x^n}{\partial z^\mu} \qquad \text{or} \qquad d\xi^\alpha = dz^\mu\frac{\partial\xi^\alpha}{\partial z^\mu} \tag{28}$$

instead of

$$dx^n = \frac{\partial x^n}{\partial z^\mu}\, dz^\mu \qquad \text{or} \qquad d\xi^\alpha = \frac{\partial\xi^\alpha}{\partial z^\mu}\, dz^\mu$$

Then, according to (1) and (5), expression (24) leads to the identity

$$d\tau^2 = dz^\mu\, dz^\nu\, \hat{g}_{\mu\nu}(z) \equiv g_{nm}(x)\, dx^n\, dx^m$$

This means that in the specific form of the product (28), the transformation (1) does not change the value of the differential

$$dx^n \Rightarrow dz^\mu\, \partial x^n/\partial z^\mu = dx^n$$

for the given concrete form of definition (5).

4.2. Transformation of the Affine Connection

It is well-known that apart from trivial tensor quantities and densities in physical laws nontensor values may appear, among which the affine connection plays an important role in the gravitational theory. Now we separate its nonhomogeneous—non-tensor term. By definition

$$\hat{\Gamma}^{\lambda}_{\nu\mu}(z)=\frac{\partial^2\xi^\alpha}{\partial z^\nu\,\partial z^\mu}\,\frac{\partial z^\lambda}{\partial \xi^\alpha}=\frac{\partial}{\partial z^\nu}\left(\frac{\partial x^\sigma}{\partial z^\mu}\,\frac{\partial \xi^\alpha}{\partial x^\sigma}\right)\frac{\partial z^\lambda}{\partial x^\rho}\,\frac{\partial x^\rho}{\partial \xi^\alpha}$$

$$=\frac{\partial x^\delta}{\partial z^\nu}\left(\frac{\partial^2 x^\sigma}{\partial x^\delta\,\partial z^\mu}\,\frac{\partial \xi^\alpha}{\partial x^\sigma}+\frac{\partial x^\sigma}{\partial z^\mu}\,\frac{\partial^2\xi^\alpha}{\partial x^\delta\,\partial x^\sigma}\right)\frac{\partial z^\lambda}{\partial x^\rho}\,\frac{\partial x^\rho}{\partial \xi^\alpha}$$

Taking into account definition (20b), we find

$$\hat{\Gamma}^{\lambda}_{\nu\mu}(z)=\frac{\partial x^\delta}{\partial z^\nu}\,\frac{\partial x^\sigma}{\partial z^\mu}\,\frac{\partial z^\lambda}{\partial x^\rho}\Gamma^{\rho}_{\delta\sigma}(x)+\frac{\partial^2 x^\rho}{\partial z^\nu\,\partial z^\mu}\,\frac{\partial z^\lambda}{\partial x^\rho} \tag{29}$$

Here the last term makes $\hat{\Gamma}^{\lambda}_{\nu\mu}(z)$ nontensor value exactly.

Tensor analysis permits us to establish a simple connection between $\hat{\Gamma}^{\lambda}_{\nu\mu}(z)$ and $\hat{g}_{\mu\nu}(z)$. Notice that

$$\frac{\partial \hat{g}_{\mu\nu}(z)}{\partial z^\lambda}=\frac{\partial}{\partial z^\lambda}\left(\eta_{\alpha\beta}\frac{\partial \xi^\alpha}{\partial z^\mu}\,\frac{\partial \xi^\beta}{\partial z^\nu}\right)$$

$$=\frac{\partial^2\xi^\alpha}{\partial z^\lambda\,\partial z^\mu}\,\frac{\partial \xi^\beta}{\partial z^\nu}\eta_{\alpha\beta}+\frac{\partial x^\rho}{\partial z^\lambda}\left(\frac{\partial \xi^\alpha}{\partial z^\mu}\,\frac{\partial^2\xi^\beta}{\partial x^\rho\,\partial z^\nu}\eta_{\alpha\beta}\right) \tag{30}$$

Further, making use of the commutator

$$\left[\frac{\partial x^\rho}{\partial z^\lambda},\frac{\partial \xi^\alpha}{\partial z^\mu}\right]_-=L^2 I^{\rho\eta}_{\lambda\mu}\frac{\partial \xi^\alpha}{\partial x^\eta}$$

we have

$$\frac{\partial \hat{g}_{\mu\nu}(z)}{\partial z^\lambda}=\frac{\partial^2\xi^\alpha}{\partial z^\lambda\,\partial z^\mu}\,\frac{\partial \xi^\beta}{\partial z^\nu}\eta_{\alpha\beta}+\frac{\partial \xi^\alpha}{\partial z^\mu}\,\frac{\partial^2\xi^\beta}{\partial z^\lambda\,\partial z^\nu}\eta_{\alpha\beta}+L^2 I^{\rho n}_{\lambda\mu}\Gamma^{\delta}_{\rho\nu}(x)g_{n\delta}(x) \tag{31}$$

On the other hand, multiplying equation (23) by $\partial\xi^\beta/\partial z^\lambda$ from the right-hand side and using the rule of multiplication (4), we arrive at the differential equation for ξ^β:

$$\hat{\Gamma}^{\lambda}_{\mu\nu}(z)\frac{\partial \xi^\beta}{\partial z^\lambda}=\frac{\partial^2\xi^\beta}{\partial z^\mu\,\partial z^\nu} \tag{32}$$

In the next calculation, the value of the commutator

$$\left[\frac{\partial \xi^\alpha}{\partial z^\mu},\frac{\partial^2\xi^\beta}{\partial z^\lambda\,\partial z^\nu}\right]_-=L^2\frac{\partial \xi^\alpha}{\partial x^\delta}\,\frac{\partial \xi^\beta}{\partial x^k}\Delta^{\delta k}_{\mu\lambda\nu}(x) \tag{33a}$$

will be needed, an estimation of which was done earlier (Namsrai, 1986), where

$$\Delta^{\delta k}_{\mu\lambda\nu}(x) = I^{\delta m}_{\mu n}\Gamma^{k}_{\lambda\mu}(x) - I^{\delta m}_{\mu\lambda}\Gamma^{k}_{m\nu}(x) + \frac{\partial}{\partial x^{\lambda}} I^{\delta k}_{\mu\nu} \tag{33b}$$

Finally, making use of (32) and (33) and again taking into account definition (27), we get

$$\frac{\partial \hat{g}_{\mu\nu}(z)}{\partial z^{\lambda}} = \hat{\Gamma}^{\rho}_{\lambda\mu}(z)\hat{g}_{\rho\nu}(z) + \hat{\Gamma}^{\rho}_{\lambda\nu}(z)\hat{g}_{\rho\mu}(z) + L^2 g_{\delta k}(x)\left(I^{\delta m}_{\mu\nu}\Gamma^{k}_{\lambda m} + \frac{\partial}{\partial x^{\lambda}} I^{\delta k}_{\mu\nu}\right) \tag{34}$$

Add to (34) the analogous relation with rearranged indices μ and λ and subtract from (34) the analogous relation with rearranged indices ν and λ. As a result, it reduces to the following connection:

$$\begin{aligned} \hat{\Gamma}^{\sigma}_{\mu\lambda}(z) &= \frac{1}{2}\left(\frac{\partial \hat{g}_{\mu\nu}}{\partial z^{\lambda}} + \frac{\partial \hat{g}_{\lambda\nu}}{\partial z^{\mu}} - \frac{\partial \hat{g}_{\mu\lambda}}{\partial z^{\nu}}\right)\hat{g}^{\nu\sigma}(z) - \frac{1}{2}L^2 g^{\nu\sigma}(x) \\ &\quad \times (N_{\mu\nu\lambda} + N_{\lambda\nu\mu} - N_{\mu\lambda\nu}) \\ N_{\mu\nu\lambda} &= I^{\delta k}_{\mu\nu}\Gamma^{\rho}_{\delta k}(x) g_{\rho\lambda}(x) + g_{\delta k}\left[I^{\delta m}_{\mu\nu}\Gamma^{k}_{\lambda m}(x) + \frac{\partial}{\partial x^{\lambda}} I^{\delta k}_{\mu\nu}\right] \end{aligned} \tag{35}$$

Here we have used definition (12b) and the expressions for the antisymmetric parts of $\hat{\Gamma}^{\rho}_{\lambda\nu}(z)$ and $\hat{g}_{\rho\nu}(z)$:

$$\hat{\Gamma}^{\rho}_{\lambda\nu} - \hat{\Gamma}^{\rho}_{\nu\lambda} = L^2 I^{\delta k}_{\lambda\nu}\Gamma^{\rho}_{\delta k}(x)$$

$$\hat{g}_{\rho\nu}(z) - \hat{g}_{\nu\rho}(z) = L^2 I^{nm}_{\rho\nu} g_{nm}(x)$$

respectively. Relation (35) will be used below in the definition of the curvature tensor in quantum space-time.

Now we give another expression for the nonhomogeneous term in the transformation rule of $\hat{\Gamma}^{\lambda}_{\nu\mu}(z)$. Differentiate the identity

$$\frac{\partial x^{\rho}}{\partial z^{\mu}}\frac{\partial z^{\lambda}}{\partial x^{\rho}} = \delta^{\lambda}_{\mu}$$

with respect to z^{ν}, from which it follows immediately that

$$\frac{\partial^2 x^{\rho}}{\partial z^{\nu}\,\partial z^{\mu}}\frac{\partial z^{\lambda}}{\partial x^{\rho}} = -\frac{\partial x^{n}}{\partial z^{\nu}}\frac{\partial x^{m}}{\partial z^{\mu}}\frac{\partial^2 z^{\lambda}}{\partial x^{n}\,\partial x^{m}}$$

Therefore, expression (29) may be written in the form

$$\hat{\Gamma}^{\lambda}_{\nu\mu}(z) = \frac{\partial x^{\delta}}{\partial z^{\nu}}\frac{\partial x^{\sigma}}{\partial z^{\mu}}\frac{\partial z^{\lambda}}{\partial x^{\rho}}\Gamma^{\rho}_{\delta\sigma}(x) - \frac{\partial x^{n}}{\partial z^{\nu}}\frac{\partial x^{m}}{\partial z^{\mu}}\frac{\partial^2 z^{\lambda}}{\partial x^{n}\,\partial x^{m}} \tag{36}$$

Now we are able to use the general covariance principle in order to prove that a "freely" falling particle satisfies the following equation of motion:

$$\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu_{\nu\lambda}(x)\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau}=\frac{1}{m}f^\mu(x) \tag{37}$$

where the proper time $d\tau^2$ is given by the formula

$$d\tau^2=g_{nm}(x)\,dx^n\,dx^m \tag{38}$$

First notice that equations (37) and (38) are valid in the absence of gravity, since for $\Gamma^\mu_{\nu\lambda}(x)=0$ and $g_{\mu\nu}=\eta_{\mu\nu}$

$$\frac{d^2x^\mu}{d\tau^2}=\frac{1}{m}f^\mu(x), \qquad d\tau^2=\eta_{\mu\nu}\,dx^\mu\,dx^\nu$$

But this coincides with equations that describe a "free" particle in the special theory of relativity modified in accordance with our assumption. Further, notice that (37) and (38) are invariant under a quantum transformation of coordinates, since

$$\frac{d^2z^\mu}{d\tau^2}=\frac{d}{d\tau}\left(\frac{dx^\nu}{d\tau}\frac{\partial z^\mu}{\partial x^\nu}\right)=\frac{d^2x^\nu}{d\tau^2}\frac{\partial z^\mu}{\partial x^\nu}+\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau}\frac{\partial^2 z^\mu}{\partial x^\mu\,\partial x^\nu}$$

whereas relation (36) leads to

$$\frac{dz^\sigma}{d\tau}\frac{dz^\beta}{d\tau}\hat{\Gamma}^\mu_{\beta\sigma}(z)=\frac{dx^\delta}{d\tau}\frac{dx^\sigma}{d\tau}\frac{\partial z^\mu}{\partial x^\rho}\Gamma^\rho_{\delta\sigma}(x)-\frac{dx^n}{d\tau}\frac{dx^m}{d\tau}\frac{\partial^2 z^\mu}{\partial x^n\,\partial x^m}$$

where we have used definition (4). Adding these two equations, we find that the left part of equation (37) is a vector, i.e.,

$$\frac{d^2z^\mu}{\partial\tau^2}+\frac{dz^\sigma}{d\tau}\frac{dz^\rho}{d\tau}\hat{\Gamma}^\mu_{\rho\sigma}(z)=\frac{\partial z^\mu}{\partial x^\nu}\left[\frac{d^2x^\nu}{d\tau^2}+\frac{dx^n}{d\tau}\frac{dx^m}{d\tau}\Gamma^\nu_{nm}(x)\right] \tag{39}$$

Thus, equation (37) as well as (38) turns out to be exactly covariant in quantum space-time. The general covariance principle tells us that relations (37) and (38) are valid in arbitrary gravitational fields, since they are indeed satisfied in quasilocal inertial systems of references. Moreover, we recall the analogous situation, which asserts that relations are valid in all systems of references (including quantum ones) if they are valid in any one system.

5. COVARIANT DIFFERENTIATION

Generally speaking, differentiation of a tensor does not lead to a new tensor, especially with respect to noncommuting variables. Nevertheless, in our scheme there exists a rule that guarantees the conservation of the tensor property of quantities after taking the differentiation operation with respect to "quantum" variables z^μ. This is just the method of generalized covariant differentiation for the usual case. Now we turn to the definition of covariant differentiation by using the affine connection $\hat{\Gamma}^\lambda_{\nu\mu}(z)$. Consider the contravariant vector $\hat{V}^\mu(z)$, the transformation rule of which is

$$\hat{V}^\mu(z) = \frac{\partial z^\mu}{\partial x^\nu} V^\nu(x)$$

Differentiation of this equality with respect to z^λ gives

$$\frac{\partial \hat{V}^\mu(z)}{\partial z^\lambda} = V^\nu(x)\frac{\partial x^n}{\partial z^\lambda}\frac{\partial^2 z^\mu}{\partial x^n \partial x^\nu} + \frac{\partial x^n}{\partial z^\lambda}\frac{\partial z^\mu}{\partial x^\nu}\frac{\partial V^\nu(x)}{\partial x^n} \tag{40}$$

The second term on the right-hand side of this equation coincides with one that would have arisen if the expression $\partial V^\mu/\partial x^\lambda$ were a tensor, but the first term breaks the tensor character of $\partial \hat{V}^\mu(z)/\partial z^\nu$. Although $\partial \hat{V}^\mu/\partial z^\lambda$ is not a tensor, by means of it one can construct a tensor. Using equation (36), we find

$$\begin{aligned}\hat{V}^\rho(z)\hat{\Gamma}^\mu_{\rho\lambda}(z) &= \frac{\partial z^\rho}{\partial x^n} V^n\left[\frac{\partial x^\delta}{\partial z^\rho}\frac{\partial x^\sigma}{\partial z^\lambda}\frac{\partial z^\mu}{\partial x^m}\Gamma^m_{\delta\sigma}(x) - \frac{\partial x^k}{\partial z^\rho}\frac{\partial x^m}{\partial z^\lambda}\frac{\partial^2 z^\mu}{\partial x^k \partial x^m}\right]\\ &= V^\sigma(x)\frac{\partial x^q}{\partial z^\lambda}\frac{\partial z^\mu}{\partial x^\rho}\Gamma^\rho_{\sigma q}(x) - V^n(x)\frac{\partial x^\rho}{\partial z^\lambda}\frac{\partial^2 z^\mu}{\partial x^n \partial x^\rho}\end{aligned} \tag{41}$$

where definition (4) has been used. Adding (40) and (41), we see that nonhomogeneous terms cancel and the result reads

$$\frac{\partial \hat{V}^\mu(z)}{\partial z^\lambda} + \hat{V}^\varkappa(z)\hat{\Gamma}^\mu_{\varkappa\lambda}(z) = \frac{\partial x^\delta}{\partial z^\lambda}\frac{\partial z^\mu}{\partial x^\nu}\left[\frac{\partial V^\nu(x)}{\partial x^\delta} + \Gamma^\nu_{\delta\sigma}(x)V^\sigma(x)\right] \tag{42}$$

Thus, we arrive at the definition of the covariant derivative in the quantum space-time case,

$$\hat{V}^\mu(z)_{;\lambda} \equiv \frac{\partial \hat{V}^\mu(z)}{\partial z^\lambda} + \hat{V}^\varkappa(z)\hat{\Gamma}^\mu_{\varkappa\lambda}(z) \tag{43}$$

and equation (42) tells us that $\hat{V}^\mu(z)_{;\lambda}$ is a tensor, since

$$\hat{V}^\mu(z)_{;\lambda} = \frac{\partial x^q}{\partial z^\lambda}\frac{\partial z^\mu}{\partial x^\nu}V^\nu(x)_{;q} \tag{44}$$

We can also define the covariant derivative of a covariant vector $U_\mu(z)$. Recall the rule of transformation

$$\hat{U}_\mu(z) = \frac{\partial x^\rho}{\partial z^\mu} U_\rho(x)$$

Differentiating this relation with respect to z^ν, we get

$$\frac{\partial \hat{U}_\mu(z)}{\partial z^\nu} = \frac{\partial^2 x^\rho}{\partial z^\nu \partial z^\mu} U_\rho(x) + \frac{\partial x^q}{\partial z^\nu} \frac{\partial x^\rho}{\partial z^\mu} \frac{\partial U_\rho(x)}{\partial x^q} \tag{45}$$

Further, from (29) it follows that

$$\hat{\Gamma}^\lambda_{\nu\mu}(z)\hat{U}_\lambda(z) = \left[\frac{\partial^2 x^\rho}{\partial z^\nu \partial z^\mu} \frac{\partial z^\lambda}{\partial x^\rho} + \Gamma^\rho_{q\sigma}(x) \frac{\partial x^q}{\partial z^\nu} \frac{\partial x^\sigma}{\partial z^\mu} \frac{\partial z^\lambda}{\partial x^\rho}\right] \frac{\partial x^n}{\partial z^\lambda} U_n(x)$$

$$= \frac{\partial^2 x^\rho}{\partial z^\nu \partial z^\mu} U_\rho(x) + \Gamma^\rho_{q\sigma}(x) \frac{\partial x^q}{\partial z^\nu} \frac{\partial x^\sigma}{\partial z^\mu} U_\rho(x) \tag{46}$$

By subtracting (46) from (45) the nonhomogeneous terms cancel and we obtain

$$\frac{\partial \hat{U}_\mu(z)}{\partial z^\nu} - \hat{\Gamma}^\lambda_{\nu\mu}(z)\hat{U}_\lambda(z) = \frac{\partial x^q}{\partial z^\nu} \frac{\partial x^\rho}{\partial z^\mu}\left[\frac{\partial U_\rho(x)}{\partial x^q} - \Gamma^\varkappa_{\rho q}(x) U_\varkappa(x)\right] \tag{47}$$

Thus, we have introduced a definition of the covariant derivative of the covariant vector

$$\hat{U}_\mu(z)_{;\nu} = \frac{\partial \hat{U}_\mu(z)}{\partial z^\nu} - \hat{\Gamma}^\lambda_{\nu\mu}(z)\hat{U}_\lambda(z) \tag{48}$$

and expression (47) tells us that $\hat{U}_\mu(z)_{;\nu}$ is a tensor, since

$$\hat{U}_\mu(z)_{;\nu} = \frac{\partial x^\rho}{\partial z^\nu} \frac{\partial x^\sigma}{\partial z^\mu} U_\sigma(x)_{;\rho}$$

Extension of the given method to the case of a general form of tensors encounters some difficulty caused by the noncommuting character of the variables z^μ. Here we give a formal procedure of taking the covariant derivative of simple tensor quantities: $\hat{T}^\nu_{1\mu}(z) = \hat{U}_\mu(z)\hat{V}^\nu(z)$, $\hat{T}^\nu_{2\mu}(z) = \hat{V}^\nu(z)\hat{U}_\mu(z)$, $\hat{T}_{\mu\nu}(z) = \hat{U}_\mu(z)\hat{V}_\nu(z)$, etc. So, by definition,

$$[\hat{T}^\mu_{1\mu}(z)]^z_{;\lambda} \overset{Df}{=} \frac{\partial x^q}{\partial z^\lambda}[T^\nu_{1\mu}(z)]^x_{;q}$$

$$= \frac{\partial x^q}{\partial z^\lambda}[\hat{U}_\mu(z)^x_{;q}\hat{V}^\nu(z) + \hat{U}_\mu(z)\hat{V}^\nu(z)^x_{;q}]$$

$$= \hat{U}_\mu(z)_{;\lambda}\hat{V}^\nu(z) + \frac{\partial x^q}{\partial z^\lambda}\hat{U}_\mu(z)\hat{V}^\nu(z)^x_{;q} \tag{49}$$

where the symbols $[\;]^z_{;q}$ and $[\;]^x_{;q}$ mean covariant differentiation with respect to variables z^λ and x^λ, respectively. Further, define the commutator

$[\partial x^q/\partial z^\lambda, \hat{U}_\mu(z)]_-$. From the definition $\hat{U}_\mu(z) = (\partial x^\tau/\partial z^\mu)U_\tau(x)$, it follows immediately that

$$\frac{\partial x^q}{\partial z^\lambda}\hat{U}_\mu(z) = U_\tau(x)\frac{\partial x^q}{\partial z^\lambda}\frac{\partial x^\tau}{\partial z^\mu}$$
$$= U_\tau\left(\frac{\partial x^\tau}{\partial z^\mu}\frac{\partial x^q}{\partial z^\lambda} + L^2 I^{q\tau}_{\lambda\mu}\right)$$
$$= \hat{U}_\mu(z)\frac{\partial x^q}{\partial z^\lambda} + L^2 I^{q\tau}_{\lambda\mu} U_\tau(x)$$

so that

$$[\partial x^q/\partial z^\lambda, \hat{U}_\mu(z)]_- = L^2 I^{q\tau}_{\lambda\mu} U_\tau(x) \tag{50}$$

Inserting this commutator into (49), we have

$$[\hat{U}_\mu(z)\hat{V}^\nu(z)]^z_{;\lambda} = \hat{U}_\mu(z)_{;\lambda}\hat{V}^\nu(z) + \hat{U}_\mu(z)\hat{V}^\nu(z)_{;\lambda} + L^2 I^{q\tau}_{\lambda\mu} U_\tau(x)\hat{V}^\nu(z)^x_{;q}$$

Recalling the differentiation rule of the product $\hat{U}_\mu(z)\hat{V}^\nu(z)$ with respect to z^λ

$$\frac{\partial}{\partial z^\lambda}(\hat{U}_\mu\hat{V}^\nu) = \frac{\partial \hat{U}_\mu}{\partial z^\lambda}\hat{V}^\nu + \frac{\partial x^q}{\partial z^\lambda}\hat{U}_\mu(z)\frac{\partial \hat{V}^\nu}{\partial x^q}$$
$$= \frac{\partial \hat{U}_\mu}{\partial z^\lambda}\hat{V}^\nu + \hat{U}_\mu(z)\frac{\partial \hat{V}^\nu}{\partial z^\lambda} + L^2 I^{q\rho}_{\lambda\mu} U_\rho(x)\frac{\partial}{\partial x^q}V^\nu(z)$$

and the definitions (43) and (48), we get finally

$$[\hat{U}_\mu(z)\hat{V}^\nu(z)]^z_{;\lambda} = \frac{\partial}{\partial z^\lambda}[\hat{U}_\mu(z)\hat{V}^\nu(z)] - \hat{\Gamma}^\varkappa_{\lambda\mu}(z)\hat{U}_\varkappa(z)\hat{V}^\nu(z)$$
$$+ \hat{U}_\mu(z)\hat{V}^\varkappa(z)\hat{\Gamma}^\nu_{\varkappa\lambda}(z) + L^2 I^{q\rho}_{\lambda\mu} U_\rho(x)\left[\hat{V}^\nu(z)^x_{;q} - \frac{\partial \hat{V}^\nu(z)}{\partial x^q}\right]$$

or in compact form

$$\hat{T}^\nu_{1\mu}(z)_{;\lambda} = \frac{\partial}{\partial z^\lambda}\hat{T}^\nu_{1\mu}(z) - \hat{\Gamma}^\varkappa_{\lambda\mu}(z)\hat{T}^\nu_{1\varkappa}(z) + \hat{T}^\varkappa_{1\mu}(z)\hat{\Gamma}^\nu_{\varkappa\lambda}(z)$$
$$+ L^2 I^{q\rho}_{\lambda\mu} U_\rho(x)\left[\hat{V}^\nu(z)^x_{;q} - \frac{\partial \hat{V}^\nu(z)}{\partial x^q}\right] \tag{51}$$

In this way terms of the type

$$\hat{V}^\nu(z)^x_{;q} - \frac{\partial \hat{V}^\nu(z)}{\partial x^q} = \Gamma^\nu_{q\varkappa}(x)V^\varkappa(x) + O(L)$$

may be expressed to our level of accuracy. An analogous calculation for $\hat{T}^{\nu}_{2\mu}(z)=\hat{V}^{\nu}(z)\hat{U}_{\mu}(z)$ gives the following expression:

$$\hat{T}^{\nu}_{2\mu}(z)_{;\lambda}=\frac{\partial}{\partial z^{\lambda}}[\hat{V}^{\nu}(z)\hat{U}_{\mu}(z)]+\hat{V}^{\varkappa}(z)\hat{\Gamma}^{\nu}_{\varkappa\lambda}(z)\hat{U}_{\mu}(z)$$
$$-\hat{V}^{\nu}(z)\hat{\Gamma}^{\varkappa}_{\lambda\mu}(z)\hat{U}_{\varkappa}(z)+L^2 I^{q\nu}_{\lambda\tau}V^{\tau}(x)\left[\frac{\partial \hat{U}_{\mu}(z)}{\partial x^{q}}-\hat{U}_{\mu}(z)^{x}_{;q}\right] \qquad (52)$$

where

$$\frac{\partial \hat{U}_{\mu}(z)}{\partial x^{q}}-U_{\mu}(z)^{x}_{;q}=\Gamma^{\varkappa}_{\mu q}(x)U_{\varkappa}(x)+O(L)$$

Next, finding the commutator $[\hat{V}^{\varkappa}(z),\hat{\Gamma}^{\lambda}_{\varkappa\mu}(z)]_{-}$ (an analogous commutator will be given below), one can express this equality through the value $\hat{T}^{\nu}_{2\mu}$.

The differentiation rule for $\hat{T}_{\mu\nu}(z)=\hat{U}_{\mu}(z)\hat{V}_{\nu}(z)$ is useful for defining the covariant derivative of the metric tensor $\hat{g}_{\mu\nu}(z)$. In this case, the expression of the type of (51) and (52) takes the form

$$\hat{T}_{\mu\nu;\lambda}=\frac{\partial \hat{T}_{\mu\nu}(z)}{\partial z^{\lambda}}-\hat{\Gamma}^{\varkappa}_{\lambda\mu}(z)\hat{T}_{\varkappa\nu}(z)-\hat{U}_{\mu}(z)\hat{\Gamma}^{\varkappa}_{\lambda\nu}(z)\hat{V}_{\varkappa}(z)$$
$$-L^2 I^{q\rho}_{\lambda\mu}\Gamma^{\varkappa}_{\nu q}(x)T_{\rho\varkappa}(x) \qquad (53)$$

It is easily seen that if we know the commutator $[\hat{U}_{\mu}(z),\hat{\Gamma}^{\varkappa}_{\lambda\nu}(z)]_{-}$, then all terms of (53) may be expressed through $\hat{T}_{\mu\nu}$. By using relation (33a), we easily calculate this commutator. The result reads

$$[\hat{U}_{\mu}(z),\hat{\Gamma}^{\varkappa}_{\lambda\nu}(z)]_{-}=L^2[-I^{\tau\varkappa}_{\mu n}\Gamma^{n}_{\lambda\nu}(x)+\Delta^{\tau\varkappa}_{\mu\lambda\nu}]U_{\tau}(x) \qquad (54)$$

where the value of $\Delta^{\tau\varkappa}_{\mu\lambda\nu}$ is given by (33b). Substituting (54) into (53), we get

$$\hat{T}_{\mu\nu}(z)_{;\lambda}=\frac{\partial \hat{T}_{\mu\nu}}{\partial z^{\lambda}}-\hat{\Gamma}^{\varkappa}_{\lambda\mu}\hat{T}_{\varkappa\nu}-\hat{\Gamma}^{\varkappa}_{\lambda\nu}\hat{T}_{\varkappa\mu}-L^2 T_{\tau\varkappa}(x)(\Delta^{\tau\varkappa}_{\mu\lambda\nu}-I^{\tau\varkappa}_{\mu n}\Gamma^{n}_{\lambda\nu})$$
$$-L^2 I^{\tau\varkappa}_{\mu q}\Gamma^{q}_{\nu\lambda}(x)T_{\tau\varkappa}(x)+L^2 I^{\tau q}_{\mu\lambda}\Gamma^{\varkappa}_{\nu q}(x)T_{\tau\varkappa}(x)$$

Similarly, the covariant derivative of the metric tensor is given by the formula

$$\hat{g}_{\mu\nu}(z)_{;\lambda}=\frac{\partial \hat{g}_{\mu\nu}}{\partial z^{\lambda}}-\hat{\Gamma}^{\varkappa}_{\lambda\mu}\hat{g}_{\varkappa\nu}-\hat{\Gamma}^{\varkappa}_{\lambda\nu}\hat{g}_{\varkappa\mu}-L^2 g_{\tau\varkappa}(x)[\Delta^{\tau\varkappa}_{\mu\lambda\nu}-I^{\tau n}_{\mu\lambda}\Gamma^{\varkappa}_{\nu n}(x)] \qquad (55)$$

Substituting (34) into (55) and taking into account

$$I^{\tau m}_{\mu\lambda}\Gamma^{\varkappa}_{\lambda m}(x)+\frac{\partial}{\partial x^{\lambda}}I^{\tau\varkappa}_{\mu\nu}=\Delta^{\tau\varkappa}_{\mu\lambda\nu}-I^{\tau\rho}_{\mu\lambda}\Gamma^{\varkappa}_{\rho\nu}(x)$$

we obtain the very attractive result

$$\hat{g}_{\mu\nu}(z)_{;\lambda}=0 \qquad (56)$$

in quantum space-time. It is natural since it disappears in quasilocal inertial coordinates (by our terminology), where $\Gamma^{\mu}_{\nu\lambda}(x)$ and $\partial g_{\mu\nu}/\partial x^{\lambda}$ become zero and the tensor is equal to zero in one system of reference; it also becomes zero in all systems of reference, including the quantum one.

6. COVARIANT DIFFERENTIATION ALONG THE CURVE

Consider tensors $T(\tau)$ and define them along the curve $Z^{\mu}(\tau)$ in quantum space-time. Such types of tensors are momentum $P^{\mu}(\tau)$ and spin $S_{\mu}(\tau)$ of the individual particle. Of course, for such tensors it is not possible to talk about covariant differentiation over z, but we can define the covariant derivative over the invariant quantity τ by means of which the curve is parametrized.

Consider the contravariant vector $\hat{A}^{\mu}(\tau)$ transforming by the rule

$$\hat{A}^{\mu}(\tau)=\frac{\partial z^{\mu}}{\partial x^{\nu}}A^{\nu}(\tau) \tag{57}$$

where the partial derivative $\partial z^{\mu}/\partial x^{\nu}$ is calculated at $Z^{\nu}=Z^{\nu}(\tau)$, so that it depends on τ. Differentiating (57) over τ, we obtain two terms,

$$\frac{d\hat{A}^{\mu}(\tau)}{d\tau}=\frac{\partial z^{\mu}}{\partial x^{\nu}}\frac{dA^{\nu}(\tau)}{d\tau}+\frac{dx^{\lambda}}{d\tau}\frac{\partial^{2}z^{\mu}}{\partial x^{\nu}\,\partial x^{\lambda}}A^{\nu}(\tau) \tag{58}$$

Second derivatives $\partial^{2}z^{\mu}/\partial x^{\nu}\,\partial x^{\lambda}$ are similar to the term that breaks the homogeneity of the transformation rule (36) for the affine connection, so that we can define the covariant derivative along the curve $Z^{\mu}(\tau)$ as follows:

$$\frac{\hat{D}\hat{A}^{\mu}(\tau)}{D\tau}\equiv\frac{d\hat{A}^{\mu}(\tau)}{d\tau}+\frac{dz^{\lambda}}{d\tau}\hat{A}^{\nu}(\tau)\hat{\Gamma}^{\mu}_{\nu\lambda}(z) \tag{59}$$

Then expressions (36), (57), and (58) show that this quantity is a vector, since

$$\begin{aligned}\frac{\hat{D}\hat{A}^{\mu}}{D\tau}&=\frac{\partial z^{\mu}}{\partial x^{\nu}}\frac{dA^{\nu}(\tau)}{d\tau}+\frac{dx^{\lambda}}{d\tau}\frac{\partial^{2}z^{\mu}}{\partial x^{\nu}\,\partial x^{\lambda}}A^{\nu}(\tau)+\frac{dz^{\lambda}}{d\tau}\frac{\partial z^{\nu}}{\partial x^{\delta}}A^{\delta}(\tau)\\&\quad\times\left[\frac{\partial x^{q}}{\partial z^{\nu}}\frac{\partial x^{\delta}}{\partial z^{\lambda}}\frac{\partial z^{\mu}}{\partial x^{\rho}}\Gamma^{\rho}_{q\delta}(x)-\frac{\partial x^{n}}{\partial z^{\nu}}\frac{\partial x^{\rho}}{\partial z^{\lambda}}\frac{\partial^{2}z^{\mu}}{\partial x^{n}\,\partial x^{\rho}}\right]\\&=\frac{\partial z^{\mu}}{\partial x^{\nu}}\left[\frac{dA^{\nu}(\tau)}{d\tau}+\frac{dx^{\delta}}{d\tau}A^{q}(\tau)\Gamma^{\nu}_{q\delta}(x)\right]\\&=\frac{\partial z^{\mu}}{\partial x^{\nu}}\frac{DA^{\nu}(\tau)}{D\tau}\end{aligned} \tag{60}$$

The similarity of formulas (59) and (43) for the covariant derivative of the vector field is obvious.

Analogous considerations allow us to introduce the covariant derivative along curve $Z^\mu(\tau)$ for the covariant vector $B_\mu(\tau)$:

$$\frac{\hat{D}\hat{B}_\mu(\tau)}{D\tau}=\frac{d\hat{B}_\mu(\tau)}{d\tau}-\frac{dz^\nu}{d\tau}\hat{\Gamma}^\lambda_{\nu\mu}(z)\hat{B}_\lambda(\tau) \tag{61}$$

Expression (29) permits us to verify easily that the obtained value is indeed a vector,

$$\frac{\hat{D}\hat{B}_\mu(\tau)}{D\tau}=\frac{\partial x^\delta}{\partial z^\mu}\frac{DB_\delta}{D\tau} \tag{62}$$

The covariant derivative along the curve $Z^\mu(\tau)$ from an arbitrary tensor $\hat{T}(\tau)$ may be defined in the same way up to $O(L^2)$.

Finally, it should be noted that in definitions (43), (48), (55), (59), and (61) a strict order of multipliers and a definite arrangement of their tensor indices are important in the sense that any other kind of expression of the type of the ones obtained breaks the tensor structure.

7. DEFINITION OF THE CURVATURE TENSOR IN QUANTUM SYSTEM OF REFERENCE

Now we attempt to construct a tensor from the metric tensor and its first and second derivatives in the quantum system of reference. In order to do this, we recall the transformation rule of the affine connection,

$$\begin{aligned}\Gamma^\lambda_{\nu\mu}(x)&=\frac{\partial^2\xi^\beta}{\partial x^\nu\,\partial x^\mu}\frac{\partial x^\lambda}{\partial\xi^\beta}\\&=\frac{\partial}{\partial x^\nu}\left(\frac{\partial z^n}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial z^n}\right)\frac{\partial z^\rho}{\partial\xi^\beta}\frac{\partial x^\lambda}{\partial z^\rho}\\&=\frac{\partial^2 z^n}{\partial x^\nu\,\partial x^\mu}\frac{\partial x^\lambda}{\partial z^n}+\frac{\partial z^n}{\partial x^\mu}\frac{\partial z^q}{\partial x^\nu}\hat{\Gamma}^\rho_{qn}(z)\frac{\partial x^\lambda}{\partial z^\rho}\end{aligned}$$

On the right-hand side of this relation there is a nonhomogeneity damaging the tensor character of $\Gamma^\lambda_{\nu\mu}(x)$ and therefore we attempt to separate it:

$$\frac{\partial^2 z^\tau}{\partial x^\nu\,\partial x^\mu}=\Gamma^\rho_{\nu\mu}(x)\frac{\partial z^\tau}{\partial x^\rho}-\frac{\partial z^n}{\partial x^\mu}\frac{\partial z^q}{\partial x^\nu}\hat{\Gamma}^\tau_{qn}(z) \tag{63}$$

In order to avoid the left part, use the noncommutability of partial derivatives. Differentiation over $x^\varkappa$ gives

$$\begin{aligned}&\frac{\partial^3 z^\tau}{\partial x^\varkappa\,\partial x^\mu\,\partial x^\nu}\\&=\left[\frac{\partial z^\tau}{\partial x^\eta}\Gamma^\eta_{\varkappa\lambda}(x)-\frac{\partial z^\rho}{\partial x^\varkappa}\frac{\partial z^\delta}{\partial x^\lambda}\hat{\Gamma}^\tau_{\delta\rho}(z)\right]\Gamma^\lambda_{\mu\nu}(x)\end{aligned}$$

$$-\left[\frac{\partial z^{\rho}}{\partial x^{\eta}}\Gamma^{\eta}_{\varkappa\mu}(x)-\frac{\partial z^{\eta}}{\partial x^{\varkappa}}\frac{\partial z^{\xi}}{\partial x^{\mu}}\hat{\Gamma}^{\rho}_{\xi\eta}(z)\right]\frac{\partial z^{\delta}}{\partial x^{\nu}}\hat{\Gamma}^{\tau}_{\delta\rho}(z)$$

$$-\frac{\partial z^{\rho}}{\partial x^{\mu}}\left[\frac{\partial z^{\delta}}{\partial x^{\eta}}\Gamma^{\eta}_{\varkappa\nu}(x)-\frac{\partial z^{\eta}}{\partial x^{\varkappa}}\frac{\partial z^{\xi}}{\partial x^{\nu}}\hat{\Gamma}^{\delta}_{\xi\eta}(z)\right]$$

$$\times\hat{\Gamma}^{\tau}_{\delta\rho}(z)+\frac{\partial z^{\tau}}{\partial x^{\lambda}}\frac{\partial\Gamma^{\lambda}_{\nu\mu}(x)}{\partial x^{\varkappa}}-\frac{\partial z^{\rho}}{\partial x^{\mu}}\frac{\partial z^{\delta}}{\partial x^{\nu}}\frac{\partial z^{\eta}}{\partial x^{\varkappa}}\frac{\partial\hat{\Gamma}^{\tau}_{\delta\rho}(z)}{\partial z^{\eta}}$$

Making use of the identity

$$\frac{\partial z^{\delta}}{\partial x^{\nu}}\frac{\partial z^{\rho}}{\partial x^{\mu}}\hat{\Gamma}^{\tau}_{\rho\delta}(z)\equiv\frac{\partial z^{\rho}}{\partial x^{\mu}}\frac{\partial z^{\delta}}{\partial x^{\nu}}\hat{\Gamma}^{\tau}_{\delta\rho}(z)$$

and carrying out some elementary calculations of the type

$$\frac{\partial z^{\eta}}{\partial x^{\varkappa}}\frac{\partial z^{\xi}}{\partial x^{\mu}}\hat{\Gamma}^{\rho}_{\xi\eta}(z)\frac{\partial z^{\delta}}{\partial x^{\nu}}$$

$$=\frac{\partial z^{\delta}}{\partial x^{\nu}}\frac{\partial^{2}\xi^{\alpha}}{\partial x^{\varkappa}\,\partial x^{\mu}}\frac{\partial z^{\rho}}{\partial\xi^{\alpha}}+L^{2}\Gamma^{q}_{\varkappa\mu}(x)I^{\rho\delta}_{q\nu}$$

$$=\frac{\partial z^{\delta}}{\partial x^{\nu}}\frac{\partial z^{\eta}}{\partial x^{\varkappa}}\frac{\partial z^{\xi}}{\partial x^{\mu}}\Gamma^{\rho}_{\xi\eta}(z)+L^{2}\Gamma^{q}_{\varkappa\mu}(x)I^{\rho\delta}_{q\nu}$$

and collecting similar terms and rearranging some indices, we get

$$\frac{\partial^{3}z^{\tau}}{\partial x^{\varkappa}\,\partial x^{\mu}\,\partial x^{\nu}}$$

$$=\frac{\partial z^{\tau}}{\partial x^{\lambda}}\left[\frac{\partial\Gamma^{\lambda}_{\mu\nu}(x)}{\partial x^{\varkappa}}+\Gamma^{\lambda}_{\varkappa\eta}(x)\Gamma^{\eta}_{\mu\nu}(x)\right]$$

$$-\frac{\partial z^{\sigma}}{\partial x^{\lambda}}\left[\frac{\partial z^{\rho}}{\partial x^{\varkappa}}\Gamma^{\lambda}_{\mu\nu}(x)+\frac{\partial z^{\rho}}{\partial x^{\nu}}\Gamma^{\lambda}_{\varkappa\mu}(x)+\frac{\partial z^{\rho}}{\partial x^{\mu}}\Gamma^{\lambda}_{\varkappa\nu}(x)\right]\hat{\Gamma}^{\lambda}_{\rho\sigma}(z)$$

$$-\frac{\partial z^{\rho}}{\partial x^{\mu}}\frac{\partial z^{\delta}}{\partial x^{\nu}}\frac{\partial z^{\eta}}{\partial x^{\varkappa}}\left[\frac{\partial\hat{\Gamma}^{\tau}_{\delta\rho}(z)}{\partial z^{\eta}}-\hat{\Gamma}^{\lambda}_{\eta\delta}(z)\hat{\Gamma}^{\tau}_{\lambda\rho}(z)-\hat{\Gamma}^{\lambda}_{\eta\rho}\hat{\Gamma}^{\tau}_{\delta\lambda}\right]+L^{2}T^{\tau}_{\nu\mu\varkappa} \tag{64}$$

where

$$T^{\tau}_{\nu\mu\varkappa}=I^{\rho\sigma}_{\nu\mu}\Gamma^{\lambda}_{\varkappa\sigma}(x)\Gamma^{\tau}_{\rho\lambda}(x)+I^{\rho\sigma}_{q\nu}\Gamma^{q}_{\varkappa\mu}(x)\Gamma^{\tau}_{\sigma\rho}(x)$$

arises from the noncommutability property of the matrix of transformation $\partial z^\rho/\partial x^\delta$.

Rearranging indices ν and $\varkappa$ and subtracting the obtained result from (64), we see that all terms involving the product of $\Gamma(x)$ and $\hat{\Gamma}(z)$ disappear and the following expression remains:

$$0=\frac{\partial z^\tau}{\partial x^\lambda}\left[\frac{\partial\Gamma^\lambda_{\mu\nu}(x)}{\partial x^\varkappa}-\frac{\partial\Gamma^\lambda_{\mu\varkappa}(x)}{\partial x^\nu}+\Gamma^\lambda_{\varkappa\eta}\Gamma^\eta_{\mu\nu}-\Gamma^\lambda_{\nu\eta}\Gamma^\eta_{\mu\varkappa}\right]$$

$$-\frac{\partial z^\rho}{\partial x^\mu}\frac{\partial z^\delta}{\partial x^\nu}\frac{\partial z^\eta}{\partial x^\varkappa}\left[\frac{\partial\hat{\Gamma}^\tau_{\delta\rho}(z)}{\partial z^\eta}-\frac{\partial\hat{\Gamma}^\tau_{\eta\rho}(z)}{\partial z^\delta}+\hat{\Gamma}^\lambda_{\delta\rho}(z)\hat{\Gamma}^\tau_{\eta\lambda}(z)-\hat{\Gamma}^\lambda_{\eta\rho}(z)\hat{\Gamma}^\tau_{\delta\lambda}(z)\right]$$

$$+L^2\left[T^\tau_{\nu\mu\varkappa}-T^\tau_{\varkappa\mu\nu}+I^{\delta\eta}_{\varkappa\nu}\left(\frac{\partial\Gamma^\tau_{\delta\mu}(x)}{\partial x^\eta}-\Gamma^\lambda_{\eta\mu}\Gamma^\tau_{\delta\lambda}\right)\right]$$

In the second term of this equation the transformation matrices $\partial z^\rho/\partial x^\mu$ and $\partial z^\sigma/\partial x^\nu$ should be rearranged. Next the obtained expression should be multiplied by the matrices $\partial x^\nu/\partial z^n$, $\partial x^\mu/\partial z^q$, and $\partial x^\varkappa/\partial z^m$ successively from the left-hand side. The result reads

$$\hat{r}^\tau_{mqn}(z)=\frac{\partial x^\varkappa}{\partial z^m}\frac{\partial x^\mu}{\partial z^q}\frac{\partial x^\nu}{\partial z^n}\frac{\partial z^\tau}{\partial x^\lambda}R^\lambda_{\mu\nu\varkappa}(x)+L^2D^\tau_{mqn}(x) \tag{65}$$

where

$$R^\lambda_{\mu\nu\varkappa}(x)=\frac{\partial\Gamma^\lambda_{\mu\nu}(x)}{\partial x^\varkappa}-\frac{\partial\Gamma^\lambda_{\mu\varkappa}}{\partial x^\nu}+\Gamma^\eta_{\mu\nu}(x)\Gamma^\lambda_{\varkappa\eta}(x)-\Gamma^\eta_{\mu\varkappa}(x)\Gamma^\lambda_{\nu\eta}(x)$$

is the usual curvature tensor, and its generalization in the quantum system of reference is

$$\hat{r}^\tau_{mqn}(z)=\frac{\partial\hat{\Gamma}^\tau_{qn}(z)}{\partial z^m}-\frac{\partial\hat{\Gamma}^\tau_{mq}(z)}{\partial z^n}+\hat{\Gamma}^\lambda_{nq}(z)\hat{\Gamma}^\tau_{m\lambda}(z)-\hat{\Gamma}^\lambda_{mq}(z)\hat{\Gamma}^\tau_{n\lambda}(z) \tag{66}$$

and

$$D^\tau_{mqn}(x)=I^{\delta\eta}_{mn}\left[\frac{\partial\Gamma^\tau_{\delta q}(x)}{\partial x^\eta}-\Gamma^\lambda_{\eta q}\Gamma^\tau_{\delta\lambda}\right]+I^{\delta\eta}_{qn}\left[\frac{\partial\Gamma^\tau_{m\delta}(x)}{\partial x^\eta}-\Gamma^\lambda_{\delta\eta}(x)\Gamma^\tau_{m\lambda}(x)\right]$$

$$+[I^{\rho\sigma}_{ln}\Gamma^l_{mq}(x)-I^{\rho\sigma}_{lm}\Gamma^l_{nq}]\Gamma^\tau_{\sigma\rho}-I^{\rho\sigma}_{mq}\Gamma^\lambda_{n\sigma}(x)\Gamma^\tau_{\rho\lambda}(x) \tag{67}$$

From these expressions, we see that in quantum space-time the tensor structure of the curvature is broken, up to the value of the L^2 term. If we redefine the curvature by the formula

$$\hat{R}^{\tau}_{mqn}(z) = \hat{r}^{\tau}_{mqn}(z) - L^2 D^{\tau}_{mqn}(x) \tag{68}$$

then its tensor structure is achieved, since in this case the tensor transformation gives

$$\hat{R}^{\tau}_{mqn}(z) = \frac{\partial x^{\varkappa}}{\partial z^m}\frac{\partial x^{\mu}}{\partial z^q}\frac{\partial x^{\nu}}{\partial z^n}\frac{\partial z^{\tau}}{\partial x^{\lambda}} R^{\lambda}_{\mu\nu\varkappa}(x) \tag{69}$$

Sometimes it is useful to express $\hat{R}^{\tau}_{mqn}(z)$ through the second derivative of the metric tensor $\hat{g}_{\mu\nu}(z)$. For this purpose we consider its covariant version $\hat{R}_{\mu\nu\varkappa\lambda}(z) = \hat{R}^{\delta}_{\mu\nu\varkappa}(z)\hat{g}_{\delta\lambda}(z)$. Taking into account definitions (66) and (68) and relation (35), we get

$$\begin{aligned}
\hat{R}_{\mu\nu\varkappa\lambda}(z) &= \hat{R}^{\delta}_{\mu\nu\varkappa}(z)\hat{g}_{\delta\lambda}(z) \\
&= \frac{1}{2}\frac{\partial}{\partial z^{\mu}}\left[\left(\frac{\partial \hat{g}_{\nu\rho}}{\partial z^{\varkappa}} + \frac{\partial \hat{g}_{\varkappa\rho}}{\partial z^{\nu}} - \frac{\partial \hat{g}_{\nu\varkappa}}{\partial z^{\rho}}\right)\hat{g}^{\rho\delta}(z)\right]\hat{g}_{\delta\lambda}(z) \\
&\quad - \frac{1}{2}\frac{\partial}{\partial x^{\mu}}[g^{\rho\delta}(x)(N_{\nu\rho\varkappa} + N_{\varkappa\rho\nu} - N_{\nu\varkappa\rho})]g_{\delta\lambda}(x) \\
&\quad - \frac{1}{2}\frac{\partial}{\partial z^{\varkappa}}\left[\left(\frac{\partial \hat{g}_{\mu\rho}}{\partial z^{\nu}} + \frac{\partial \hat{g}_{\nu\rho}}{\partial z^{\mu}} - \frac{\partial \hat{g}_{\mu\nu}}{\partial z^{\rho}}\right)\hat{g}^{\rho\delta}(z)\right]\hat{g}_{\delta\lambda}(z) \\
&\quad + \frac{1}{2}\frac{\partial}{\partial x^{\varkappa}}[g^{\rho\delta}(x)(N_{\mu\rho\nu} + N_{\nu\rho\mu} - N_{\mu\nu\rho})]g_{\delta\lambda}(x) \\
&\quad + [\hat{\Gamma}^{\lambda}_{\varkappa\nu}(z)\hat{\Gamma}^{\sigma}_{\mu\lambda}(z) - \hat{\Gamma}^{\lambda}_{\mu\nu}(z)\hat{\Gamma}^{\sigma}_{\varkappa\lambda}(z)]\hat{g}_{\sigma\lambda}(z) \\
&\quad - L^2 D^{\sigma}_{\mu\nu\varkappa}(x)g_{\sigma\lambda}(x)
\end{aligned} \tag{70}$$

in accordance with our level of accuracy. Further, by using the identity

$$\hat{g}^{\rho\delta}(z)\hat{g}_{\delta\lambda}(z) = \delta^{\rho}_{\lambda}$$

one can easily verify that

$$\frac{\partial \hat{g}^{\rho\sigma}}{\partial z^{\varkappa}}\hat{g}_{\sigma\lambda}(z) = -\hat{g}^{\rho\sigma}(z)\frac{\partial \hat{g}_{\sigma\lambda}}{\partial z^{\varkappa}} - L^2 Q^{\rho}_{\varkappa\lambda} \tag{71}$$

where

$$Q^{\rho}_{\varkappa\lambda}=(g^{\rho l}(x)I^{\delta n}_{l\varkappa}+g^{k\delta}(x)I^{\rho n}_{k\varkappa})\frac{\partial g_{\sigma\lambda}(x)}{\partial x^{n}}-g^{k\delta}(x)I^{q\rho}_{kq}\frac{\partial g_{\sigma\lambda}(x)}{\partial x^{\varkappa}}$$

After inserting relation (34) into equality (71), the result reads

$$\frac{\partial \hat{g}^{\rho\delta}(z)}{\partial z^{\varkappa}}\hat{g}_{\delta\lambda}(z)=-\hat{g}^{\rho\sigma}(z)[\hat{\Gamma}^{\eta}_{\varkappa\sigma}(z)\hat{g}_{\eta\lambda}(z)+\hat{\Gamma}^{\eta}_{\varkappa\lambda}\hat{g}_{\eta\sigma}(z)]$$

$$-L^{2}\{g_{qk}(x)g^{\rho\sigma}(x)[I^{qm}_{\sigma\lambda}\Gamma^{k}_{\varkappa m}(x)+\frac{\partial}{\partial x^{\varkappa}}I^{qk}_{\sigma\lambda}]+Q^{\rho}_{\varkappa\lambda}\} \qquad (72)$$

Moreover, in expression (70) a transformation of the type

$$\frac{\partial}{\partial z^{\mu}}\left[\frac{\partial \hat{g}_{\nu\rho}(z)}{\partial z^{\varkappa}}\hat{g}^{\rho\delta}(z)\right]$$

$$=\frac{\partial^{2}\hat{g}_{\nu\rho}(z)}{\partial z^{\mu}\,\partial z^{\varkappa}}\hat{g}^{\rho\delta}(z)+\frac{\partial \hat{g}_{\nu\rho}(z)}{\partial z^{\varkappa}}\frac{\partial \hat{g}^{\rho\delta}}{\partial z^{\mu}}$$

$$+L^{2}\left[L^{n,\varkappa}_{\mu,\nu\rho}+I^{nm}_{\mu\varkappa}\frac{\partial}{\partial x^{m}}g_{\nu\rho}(x)\right]\frac{\partial g^{\rho\delta}(x)}{\partial x^{n}} \qquad (73)$$

should be carried out, where

$$L^{n,m}_{\mu,\nu\rho}=\frac{\partial g_{\nu\beta}}{\partial x^{m}}I^{n\beta}_{\mu\rho}+\frac{\partial g_{\alpha\rho}}{\partial x^{m}}I^{n\alpha}_{\mu\nu}+g_{\alpha\rho}(x)N^{n\alpha}_{\mu,m\nu}+g_{\nu\beta}N^{n\beta}_{\mu,m\rho}$$

and $N^{n\beta}_{\mu,m\rho}\to\Delta^{n\beta}_{\mu,m\rho}$ is given by (33b).

Finally, taking into account relations (71)-(73) and after some elementary but tedious calculations, expression (70) takes the form

$$\hat{R}_{\mu\nu\varkappa\lambda}(z)=\frac{1}{2}\left(\frac{\partial^{2}\hat{g}_{\varkappa\lambda}}{\partial z^{\mu}\,\partial z^{\nu}}+\frac{\partial^{2}\hat{g}_{\mu\nu}}{\partial z^{\varkappa}\,\partial z^{\lambda}}-\frac{\partial^{2}\hat{g}_{\nu\varkappa}}{\partial z^{\mu}\,\partial z^{\lambda}}-\frac{\partial^{2}\hat{g}_{\mu\lambda}}{\partial z^{\varkappa}\,\partial z^{\nu}}\right)$$

$$+(\hat{\Gamma}^{\delta}_{\mu\nu}\hat{\Gamma}^{\eta}_{\varkappa\lambda}-\hat{\Gamma}^{\delta}_{\nu\varkappa}\hat{\Gamma}^{\eta}_{\mu\lambda})\hat{g}_{\eta\delta}(z)+L^{2}[D^{1}_{\mu\nu\varkappa\lambda}-D^{\sigma}_{\mu\nu\varkappa}g_{\sigma\lambda}(x)] \quad (74)$$

where $D^{\sigma}_{\mu\nu\varkappa}(x)$ is given by the formula (67), and

$$D^{1}_{\mu\nu\varkappa\lambda}(x)=I^{\delta k}_{\varkappa\nu}\Gamma^{\sigma}_{\delta k}(x)\Gamma^{\eta}_{\mu\sigma}(x)g_{\eta\lambda}(x)+\frac{1}{2}I^{nm}_{\mu\varkappa}\frac{\partial^{2}g_{\nu\lambda}}{\partial x^{n}\,\partial x^{m}}-T_{\mu\nu\varkappa\lambda}+T_{\varkappa\nu\mu\lambda}$$

$$+\frac{1}{2}\frac{\partial}{\partial x^{\varkappa}}(N_{\nu\mu\lambda}-N_{\mu\nu\lambda})$$

$$T_{\mu\nu\varkappa\lambda}=\Gamma^{\sigma}_{\varkappa\nu}(x)g_{nk}(x)[I^{nm}_{\sigma\lambda}\Gamma^{k}_{\mu m}(x)+\frac{\partial}{\partial x^{\mu}}I^{nk}_{\sigma\lambda}]$$

$$+[\Gamma^{l}_{\nu\varkappa}(x)I^{\sigma n}_{l\mu}+\Gamma^{\sigma}_{m\varkappa}(x)I^{nm}_{\mu\nu}+\Gamma^{\sigma}_{m\nu}(x)I^{nm}_{\mu\varkappa}+N^{n\sigma}_{\mu,\varkappa\nu}]\frac{\partial g_{\sigma\lambda}}{\partial x^{n}}$$

$$-\Gamma^{m}_{\nu\varkappa}(x)g_{\delta m}(x)I^{q\delta}_{kq}g^{k\sigma}(x)\frac{\partial g_{\sigma\lambda}(x)}{\partial x^{\mu}}+\frac{1}{2}\frac{\partial}{\partial x^{\mu}}(N_{\nu\lambda\varkappa}+N_{\varkappa\lambda\nu}-N_{\nu\varkappa\lambda})$$

$$N_{\mu\nu\lambda}=I^{\delta k}_{\mu\nu}\Gamma^{\eta}_{\delta k}(x)g_{\eta\lambda}(x)+g_{\delta k}(x)\left[I^{\delta m}_{\mu\nu}\Gamma^{k}_{\lambda m}(x)+\frac{\partial}{\partial x^{\lambda}}I^{\delta k}_{\mu\nu}\right] \tag{75}$$

8. THE EINSTEIN EQUATION IN QUANTUM SPACE-TIME

First we note that it is difficult to reconstruct the Einstein equation in quantum space-time by using first principles as is done in the usual theory of gravity. However, if we use the general covariance principle discussed in Section 1, then the corresponding generalization of the Einstein equation may be made by redefining the Ricci tensor $R_{\varkappa\mu}(x)$, scalar curvature R, and the energy-momentum tensor $T_{\mu\nu}$ which enter into the usual Einstein equation. Now we go on to the redefinition of these quantities.

We know that $R_{\mu\varkappa}(x)=R^{\lambda}_{\mu\lambda\varkappa}$ is the Ricci tensor; then, by definition (4),

$$\hat{R}_{mq}(z)=\hat{R}^{\tau}_{mq\tau}(z) \tag{76}$$

is also a tensor in quantum space-time. We call it the generalized Ricci tensor. It is easy to verify that in our case

$$\hat{R}_{mqn\lambda}(z)=\hat{R}^{\tau}_{mqn}(z)\hat{g}_{\tau\lambda}(z)$$

is also a tensor. Indeed,

$$\hat{R}_{mqn\lambda}(z)=\frac{\partial x^{\varkappa}}{\partial z^{m}}\frac{\partial x^{\mu}}{\partial z^{q}}\frac{\partial x^{\nu}}{\partial z^{n}}\frac{\partial z^{\lambda}}{\partial x^{\eta}}R^{\eta}_{\mu\nu\varkappa}(x)g_{lk}(x)\frac{\partial x^{l}}{\partial x^{\tau}}\frac{\partial x^{k}}{\partial z^{\lambda}}$$

$$=\frac{\partial x^{\varkappa}}{\partial z^{m}}\frac{\partial x^{\mu}}{\partial z^{q}}\frac{\partial x^{\nu}}{\partial z^{n}}\frac{\partial x^{k}}{\partial z^{\lambda}}R^{\eta}_{\mu\nu\varkappa}(x)g_{\eta k}(x)$$

since $R_{k\mu\nu\varkappa}(x)=R^{\eta}_{\mu\nu\varkappa}(x)g_{\eta k}(x)$ is a tensor. However, another contraction gives a different result:

$$\hat{R}^{1}_{mq}(z)=\hat{R}_{mqn\lambda}(z)\hat{g}^{\lambda n}(z)$$

$$=\frac{\partial x^{\varkappa}}{\partial z^{m}}\frac{\partial x^{\mu}}{\partial z^{q}}g^{\beta\nu}(x)R_{\beta\mu\nu\varkappa}(x)+L^{2}g^{\beta l}I^{\delta\nu}_{l\delta}R_{\beta q\nu m}(x) \tag{77}$$

where $R_{\mu\varkappa}=g^{\beta\nu}(x)R_{\beta\mu\nu\varkappa}(x)$ is the Ricci tensor.

Further, we notice that scalar curvature in our case is given by the formula

$$\hat{R}=\hat{R}_{mqn\lambda}(z)\hat{g}^{\lambda n}(z)\hat{\rho}^{qm}(z)=R+2L^{2}g^{\mu n}(x)I^{\delta\varkappa}_{n\delta}R_{\mu\varkappa}(x) \tag{78}$$

Now the question arises of how to redefine the energy-momentum tensor in a quantum system of reference. We assume that its covariant structure is conserved under a "quantum" transformation of coordinates. Thus,

$$\hat{T}_{\mu\nu}(z)=\frac{\partial x^{\rho}}{\partial z^{\mu}}\frac{\partial x^{\delta}}{\partial z^{\nu}}T_{\rho\delta}(x)=\hat{T}^{s}_{\mu\nu}(z)+T^{A}_{\mu\nu}(z) \tag{79}$$

where

$$T^{s}_{\mu\nu}(z)=\frac{1}{2}\left(\frac{\partial x^{\rho}}{\partial z^{\mu}}\frac{\partial x^{\delta}}{\partial z^{\nu}}+\frac{\partial x^{\rho}}{\partial z^{\nu}}\frac{\partial x^{\delta}}{\partial z^{\mu}}\right)T_{\rho\delta}(x)$$

is the symmetric part of $\hat{T}_{\mu\nu}(z)$ and is responsible for conservation of matter in quantum space-time. Its antisymmetric part

$$T^{A}_{\mu\nu}(z)=\tfrac{1}{2}L^{2}I^{\delta\rho}_{\mu\nu}(x)T_{\delta\rho}(x)$$

may give rise to the generation and disappearance of matter from and into vacuum states. This is the problem of the quantum theory of gravity, discussion of which is beyond the scope of this paper. Thus, we suggest that in quantum space-time the Einstein equation takes the form

$$\hat{R}_{\mu\nu}(z)-\tfrac{1}{2}\hat{g}_{\mu\nu}(z)\hat{R}=-8\pi G\hat{T}_{\mu\nu}(z) \tag{80}$$

in accordance with our assumption. Here the quantities $\hat{R}_{\mu\nu}(z)$, $\hat{g}_{\mu\nu}(z)$, $\hat{R}$, and $\hat{T}_{\mu\nu}(z)$ are given by formulas (76), (27), (78), and (79), respectively. The solution of equation (80) is not known and needs deeper study in this direction and another fundamental physical principle.

REFERENCES

Frazer, F., Duncan, W., and Collar, A. (1952). *Elementary Matrices*, Cambridge University Press, Oxford, p. 133.

Namsrai, Kh. (1986). *International Journal of Theoretical Physics*, **25**, 1035–1074.

Weinberg, S. (1972). *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, Wiley, New York.